TWO ∞-CATEGORIES: $\mathbf{Cat}_\infty$ AND $\mathbf{An}$

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Today: ∞-category of ∞-categories and ∞-category of animae

Need: Ordinary-to-quasi-category "dictionary"

Main reference: I mostly follow [Wag25, Chapter 2].

1. Setting the stage: Quasicategories

We first recall some facts and set up the notation. We recall that every morphism in the simplicial category Δ is a composition of *faces*

$$\{d_i\colon [n-1] \to [n]\}_{n\geq 1, 0\leq i\leq n}$$

and *degeneracies*

$$\{s_j\colon [n] \to [n-1]\}_{n\geq 1, 0\leq j\leq n-1},$$

see [Mac95] for instance. The face d_i is the unique increasing injection that "misses" i and the degeneracy s_j is the unique increasing surjection that sends both j and $j+1$ to j. As usual, for a simplicial set X, we denote $X([n])$ by X_n. By Yoneda Lemma, we have

$$X_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X).$$

The Yoneda embedding gives rise to a cosimplicial simplicial set

$$よ\colon \Delta \to \mathbf{PSh}(\Delta) = \mathbf{sSet}$$

sending $[n]$ to the standard simplex $\Delta^n = \mathrm{Hom}_{\mathbf{sSet}}(-, [n])$.

Definition 1.1. Let $n \geq 0$. The *boundary* of the standard n-simplex Δ^n is the simplicial subset

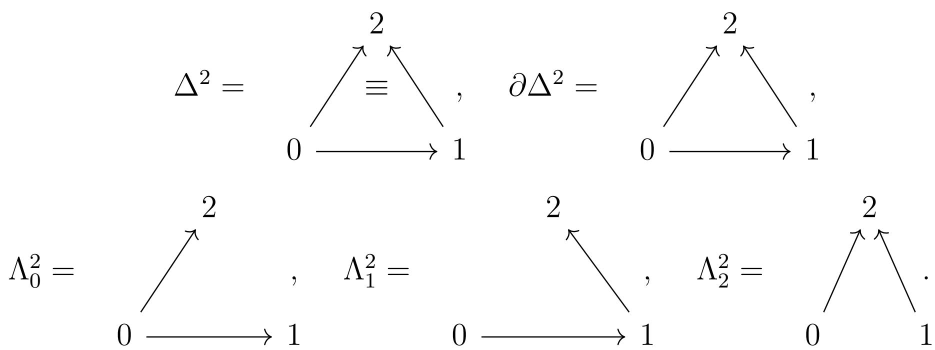
$$\partial\Delta^n := \bigcup_{i=0}^{n} \mathrm{im}(d_i\colon \Delta^{n-1} \to \Delta^n) \subset \Delta^n.$$

For $0 \leq j \leq n$, define the *j-th horn* as the simplicial subset

$$\Lambda^n_j := \bigcup_{i=0, i\neq j}^{n} \mathrm{im}(d_i\colon \Delta^{n-1} \to \Delta^n) \subset \Delta^n.$$

Vocabulary: For $0 < j < n$, we call Λ^n_j *inner* horns. For $j = 0$ and $j = n$, we call we call Λ^n_j *outer* horns.

Example 1.2. Here are some pictures for $n = 2$:



Definition 1.3 (Boardman–Vogt, [BV73])**.** A *quasicategory* is a simplicial set $\mathcal{C}$ such that for all $n \geq 2$ and $0 < i < n$, every inner horn filling problem

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

has a solution. If the above problem has a solution for all $n \geq 1$ and $0 \leq i \leq n$, we call $\mathcal{C}$ a *Kan complex*.

We denote by **Kan** and **QCat** the full subcategories of **sSet**:

$$\mathbf{Kan} \subset \mathbf{QCat} \subset \mathbf{sSet}.$$

Example 1.4. (1) If X is a topological space, then $\mathrm{Sing}(X)$ is a Kan complex and in particular a quasicategory. To see this, use the adjunction $|-|\colon \mathbf{sSet} \rightleftarrows \mathbf{Top}\colon \mathrm{Sing}(-)$ to translate the horn filling problem

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Sing}(X) \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

into the problem

$$\begin{array}{ccc} |\Lambda^n_i| & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ |\Delta^n|. & & \end{array}$$

Now there is a (deformation) retraction $|\Delta^n| \to |\Lambda^n_i|$, see [Lur18, Proposition 1.2.5.8.] for an explicit formula.

(2) Let $\mathcal{C}$ be a small category. Every inner horn filling problem

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{N}(\mathcal{C}) \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

has a **unique** solution, see [Rez22]. Moreover, each simplicial set with such property must be the nerve of some category, the homotopy category.

(3) We have $\Delta^n \cong \mathrm{N}([n])$, where $[n]$ is the poset category.

(4) In the Talk 1, we have seen the adjunction

$$- \times X \colon \mathbf{sSet} \rightleftarrows \mathbf{sSet} \colon \mathrm{F}(X, -).$$

By Yoneda lemma,

$$\mathrm{F}(X,Y)_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{F}(X,Y)).$$

We have the following nontrivial fact (see [Rez22] for a proof):

If $\mathcal{C}, \mathcal{D} \in \mathbf{QCat}$, then $F(C, D) \in \mathbf{QCat}$.

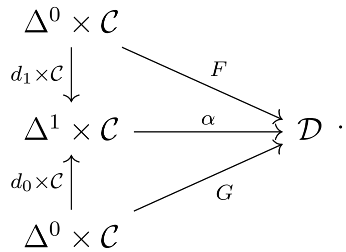
Definition 1.5. Let $\mathcal{C}$ a quasicategory. Define *objects* of $\mathcal{C}$ as the vertices, that is, the elements of the set $\mathcal{C}_0$. For $x \in \mathcal{C}_0$, we simply write $x \in \mathcal{C}$. We say that $\alpha \in \mathcal{C}_1$ is a *morphism from $x \in \mathcal{C}$ to $y \in \mathcal{C}$*, denoted by $\alpha \colon x \to y$, if $d_1^*(\alpha) = x$, $d_0^*(\alpha) = y$. For $x \in \mathcal{C}$, $s_0^*(x) \in \mathcal{C}_1$ is the identity on x, denoted by $\mathrm{id}_x \colon x \to x$.

In the above definition $d_0^*, d_1^* \colon \mathcal{C}_1 \to \mathcal{C}_0$ and $s_0^* \colon \mathcal{C}_0 \to \mathcal{C}_1$ are images under $\mathcal{C} \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ of morphisms in Δ

$$\begin{aligned} d_0 \colon [0] &\to [1], & d_1 \colon [0] &\to [1], & s_0 \colon [1] &\to [0]. \\ 0 &\mapsto 1 & 0 &\mapsto 0 & 0 &\mapsto 0 \\ & & & & 0 &\mapsto 1. \end{aligned}$$

For $x \in \mathcal{C}_0$, we will write $\{x\} \to \mathcal{C}$ for the morphism of simplicial sets $\Delta^0 \to \mathcal{C}$, which sends $f \in \Delta^0_k$ to $f^*(x) \in \mathcal{C}_k$.

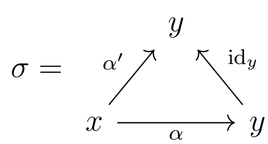
Example 1.6. Let $\mathcal{C}, \mathcal{D} \in \mathbf{QCat}$. The set of objects of the quasicategory $\mathrm{F}(\mathcal{C}, \mathcal{D})$ (see Example 1.4) is given by $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^0 \times \mathcal{C}, \mathcal{D})$. Its elements are called *functors* from $\mathcal{C}$ to $\mathcal{D}$. Now let $F, G \in \mathrm{F}(\mathcal{C}, \mathcal{D})_0$. To give a morphism $\alpha \in \mathrm{F}(\mathcal{C}, \mathcal{D})_1$ from F to G, means to give a natural transformation $\Delta^1 \times \mathcal{C} \to \mathcal{D}$ such that $d_1^*(\alpha) = F$ and $d_0^*(\alpha) = G$. In other words, the following diagram commutes:



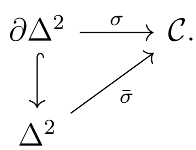
We rightly call such α a *homotopy* from F to G.

Now, *how does one compose morphisms in a quasicategory*?

Definition 1.7. Let $\mathcal{C}$ be a quasicategory. Two morphisms $\alpha, \alpha' \colon x \to y$ are said to be *equivalent*, denoted $\alpha \simeq \alpha'$, if the map $\sigma \colon \partial\Delta^2 \to \mathcal{C}$

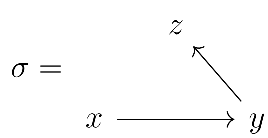


extends to a map $\bar{\sigma} \colon \Delta^2 \to \mathcal{C}$ such that the following diagram commutes



Exercise 1.8. This is an equivalence relation on $\mathcal{C}_1$.

Definition 1.9. Let $\mathcal{C}$ be a quasicategory. For $\alpha \colon x \to y$, $\beta \colon y \to z$, fill the horn $\sigma \colon \Lambda^2_1 \to \mathcal{C}$



with $\bar{\sigma} \colon \Delta^2 \to \mathcal{C}$ so that $\bar{\sigma}|_{\Lambda^2_1} = \sigma$. Any $\gamma \colon x \to z$ represented by $\bar{\sigma}|_{\Delta^{\{0,2\}}}$ is called a *composition* of α and β. We write $\gamma \simeq \beta \circ \alpha$.

Remark 1.10. Even though it looks clear from the pictures, let us explain what the above $\Delta^{\{0,2\}}$ means. More generally, take any finite totally ordered set S. Define $(\Delta^S)_k = \{\text{order preserving } [k] \to S\}$. Note that there exists a unique $n \in \mathbb{N}$ and a unique order preserving bijection $[n] \to S$. Hence one obtains a unique iso $\Delta^S \cong \Delta^n$.

Exercise 1.11. If γ and γ' are compositions of α and β, then $\gamma \simeq \gamma'$.

Definition 1.12. The homotopy category of a quasicategory $\mathcal{C}$ is a category whose objects are given by objects of $\mathcal{C}$ and whose morphisms are given by $\mathcal{C}_1$ modulo $\simeq$.

Example 1.13. (1) If X is a topological space, then

$$h\mathrm{Sing}(X) \cong \pi_1(X),$$

the fundamental groupoid of X.

(2) If $\mathcal{C}$ is a small category, then $hN(\mathcal{C}) \cong \mathcal{C}$.

We finish this part with equivalences and animae:

Definition 1.14. A morphism $\alpha\colon x \to y$ in a quasicategory $\mathcal{C}$ is an *equivalence* if the horns $\Lambda^2_0 \to \mathcal{C}$, respectively $\Lambda^2_2 \to \mathcal{C}$,

$$\begin{array}{ccc} & x & \\ {}^{\mathrm{id}_x}\nearrow & & \\ x & \xrightarrow{\alpha} & y \end{array} \qquad\qquad \begin{array}{ccc} & y & \\ {}^{\mathrm{id}_y}\nearrow & & \nwarrow^{\alpha} \\ y & & x \end{array}$$

can be filled. A quasicategory is an *anima* if all its morphisms are equivalences.

Theorem 1.15 (Joyal, [Joy02])**.** *A quasicategory $\mathcal{C}$ is a Kan complex if and only if it is an anima.*

Note that every Kan complex is an anima since the horns in the above definition are the outer horns. The converse in nontrivial. In view of the above theorem and agreement in the notes I follow [Wag25], we use the following **terminology**:

$$\text{Kan complexes} \equiv \text{animae}.$$

Example 1.16. Let $\mathcal{C}$ be a quasicategory and $\mathrm{Core}(\mathcal{C})$ the subquasicategory spanned by equivalences. Namely, the subset

$$S_1 = \{\text{equivalences}\} \subset \mathcal{C}_1$$

is closed under $\simeq$ and $\circ$. We say that $\Delta^n \to \mathcal{C}$ (think Yoneda!) is in $\mathrm{Core}(\mathcal{C})$ iff for all $0 \leq i \leq j \leq n$,

$$\Delta^{\{i,j\}} \to \Delta^n \to \mathcal{C}$$

lies in S_1. Observe that taking core commutes with the nerve construction. As an exercise, one can show that $\mathrm{Core}(\mathcal{C})$ is the largest anima contained in $\mathcal{C}$.

2. Homotopy coherent nerve

Definition 2.1 (see [Lan21] for a complete definition)**.** A simplicially enriched category $\mathcal{C}$ consists of a set of objects such that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ are simplicial sets, denoted $\mathrm{F}_{\mathcal{C}}(x, y)$ for $x, y \in \mathcal{C}$, the composition

$$\mathrm{F}_{\mathcal{C}}(x, y) \times \mathrm{F}_{\mathcal{C}}(y, z) \to \mathrm{F}_{\mathcal{C}}(x, z)$$

is a morphism of simplicial sets, and so that $id_x \colon x \to x$ is a 0-simplex, i.e. $\mathrm{id}_x \in \mathrm{F}_{\mathcal{C}}(x, x)_0$, which satisfy some axioms.

For simplicially enriched categories, there is a notion of a simplicially enriched functor, see [Lan21]. We denote the category of simplicially enriched categories and simplicially enriched functors by $\mathbf{Cat}_{\Delta}$.

Lemma 2.2. Cat_{Δ} *has colimits.*

For a proof, see [Lan21]. In the Talk 1, the fact that **Cat** has colimits allowed us to construct the adjunction $h \colon \mathbf{sSet} \rightleftarrows \mathbf{Cat} \colon \mathrm{N}(-)$:

$$\begin{array}{ccc} \Delta & \xrightarrow{U} & \mathrm{Cat} \\ {\scriptstyle \text{よ}}\downarrow & \overset{h}{\nearrow} \underset{\mathrm{N}}{\swarrow} & \\ \mathrm{sSet} & & \end{array} .$$

Here $U([n]) = [n]$ and $U(f) = f$. In this talk, we construct the simplicial nerve $\mathrm{N}^{\Delta} \colon \mathbf{Cat}_{\Delta} \to \mathbf{sSet}$, a.k.a. the homotopy coherent nerve:

$$\begin{array}{ccc} \Delta & \xrightarrow{\mathfrak{C}[-]} & \mathrm{Cat}_{\Delta} \\ {\scriptstyle \text{よ}}\downarrow & \nearrow \underset{\mathrm{N}^{\Delta}}{\swarrow} & \\ \mathrm{sSet} & & \end{array} .$$

Let us give a brief outline of the construction of $\mathfrak{C}[-]$ by Cordier and Porter [CP86]. The functor $\mathfrak{C}[-]$ sends $[n]$ to the simplicially enriched category $\mathfrak{C}[\Delta^n]$. Objects of $\mathfrak{C}[\Delta^n]$ are again nonnegative integers $0, \ldots, n$. Hom simplicial sets are given by formula

$$\mathrm{F}_{\mathfrak{C}[\Delta^n]}(i, j) = \begin{cases} \emptyset & \text{if } i > j, \\ \Delta^0 & \text{if } i = j, \\ \square^{j-i-1}, & \text{if} i < j, \end{cases}$$

where $\square^n := (\Delta^1)^{\times n}$. For $i < j < k$,

$$\mathrm{F}_{\mathfrak{C}[\Delta^n]}(i, j) \times \mathrm{F}_{\mathfrak{C}[\Delta^n]}(j, k) \to \mathrm{F}_{\mathfrak{C}[\Delta^n]}(i, k)$$

is defined by

$$\square^{j-i-1} \times \square^{k-j-1} \xrightarrow{\cong} \square^{j-i-1} \times \{1\} \times \square^{k-j-1} \subset \square^{k-i-1}.$$

We do not spell here the other cases and what $\mathfrak{C}[-]$ does on morphisms.

Remark 2.3. In [CP86] and references therein is provided a bit more conceptual explanation of construction $\mathfrak{C}[-]$. See also a blog post [Rie] and an expository note [Rie23].

Definition 2.4. The *simplicial nerve* (or the *homotopy coherent nerve*) $\mathrm{N}^\Delta \colon \mathbf{Cat}_\Delta \to \mathbf{sSet}$ is defined by setting for all $\mathcal{C} \in \mathbf{Cat}_\Delta$ and all $n \in \mathbb{N}$,

$$\mathrm{N}^\Delta(\mathcal{C})_n = \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Theorem 2.5 (Cordier–Porter, [CP86])**.** *Let $\mathcal{C}$ be a small simplicially enriched category. If $\mathcal{C}$ is Kan enriched (that is, if Hom simplicial sets are Kan complexes), then $\mathrm{N}^\Delta(\mathcal{C})$ is a quasicategory.*

3. Main examples

Simplicial sets. Following Example 1.4, one can enrich $\mathbf{sSet}$ to obtain $\mathbf{sSet}^\Delta \in \mathbf{Cat}_\Delta$: set

$$\mathrm{F}_{\mathbf{sSet}^\Delta}(X, Y) = \mathrm{F}(X, Y).$$

Kan complexes. Example 1.4 is even more optimistic: it is a fact that $\mathrm{F}(X, Y)$ is a Kan complex, whenever Y is a Kan complex. Thus, one can enrich $\mathbf{Kan}$ to obtain $\mathbf{Kan}^\Delta \in \mathbf{Cat}_\Delta$: again, set

$$\mathrm{F}_{\mathbf{Kan}^\Delta}(X, Y) = \mathrm{F}(X, Y).$$

By an application of Theorem 2.5 (modulo set-theoretic issues), one defines the *quasicategory of animae*

$$\mathbf{An} := \mathrm{N}^\Delta(\mathbf{Kan}^\Delta).$$

Quasicategories. We can also turn $\mathbf{QCat}$ into a simplicially enriched category $\mathbf{QCat}^\Delta \in \mathbf{Cat}_\Delta$ by setting

$$\mathrm{F}_{\mathbf{QCat}^\Delta}(\mathcal{C}, \mathcal{D}) = \mathrm{Core}\ \mathrm{F}(\mathcal{C}, \mathcal{D}).$$

Since $\mathrm{Core}\ \mathrm{F}(\mathcal{C}, \mathcal{D})$ is the largest anima contained in $\mathrm{F}(\mathcal{C}, \mathcal{D})$, it is a Kan complex by Theorem 1.15. In this way $\mathbf{QCat}^\Delta$ is in fact Kan enriched. Again, using Theorem 2.5, one defines the *quasicategory of quasicategories*

$$\mathbf{Cat}_\infty := \mathrm{N}^\Delta(\mathbf{QCat}^\Delta).$$

Understanding low-dimensional simplices. We first have

$$(\mathbf{Cat}_\infty)_0 = \mathrm{N}^\Delta(\mathbf{QCat}^\Delta)_0 = \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\mathfrak{C}[\Delta^0], \mathcal{C}),$$

so 0-simplices are just quasicategories: $\mathrm{F}_{\mathfrak{C}[\Delta^0]}(0,0)$ contains only id_0.

Next,

$$(\mathbf{Cat}_\infty)_1 = \mathrm{N}^\Delta(\mathbf{QCat}^\Delta)_1 = \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\mathfrak{C}[\Delta^1], \mathcal{C}),$$

so 1-simplices are functors between quasicategories. Indeed, say 0 resp. 1 are sent to quasicategories $\mathcal{C}$ resp. $\mathcal{D}$. We should specify

$$\begin{array}{ccc} \mathrm{F}_{\mathfrak{C}[\Delta^1]}(0,1) & \to & \mathrm{F}_{\mathbf{QCat}^\Delta}(\mathcal{C},\mathcal{D}). \\ = \Delta^0 & & = \mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{D}) \end{array}$$

But, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{D})) &\cong (\mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{D}))_0 \\ &= \mathrm{F}(\mathcal{C},\mathcal{D})_0 \\ &= \{\text{functors from } \mathcal{C} \text{ to } \mathcal{D}\}. \end{aligned}$$

Finally,

$$(\mathbf{Cat}_\infty)_2 = \mathrm{N}^\Delta(\mathbf{QCat}^\Delta)_2 = \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\mathfrak{C}[\Delta^2], \mathcal{C}).$$

Hence a 2-simplex $\sigma\colon \Delta^2 \to \mathbf{Cat}_\infty$ corresponds to a simplicially enriched functor $\bar{\sigma}\colon \mathfrak{C}[\Delta^2] \to \mathbf{QCat}^\Delta$. Suppose $0, 1$, and 2 are sent to $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ respectively, so that

$$\begin{aligned} \sigma\big|_{\Delta^{\{0,1\}}} &= d_2^*(\sigma) = F \in \mathrm{F}(\mathcal{C},\mathcal{D})_0 \quad \text{and} \\ \sigma\big|_{\Delta^{\{1,2\}}} &= d_0^*(\sigma) = G \in \mathrm{F}(\mathcal{D},\mathcal{E})_0. \end{aligned}$$

Now we need to specify the map

$$\begin{array}{ccc} \mathrm{F}_{\mathfrak{C}[\Delta^1]}(0,2) & \to & \mathrm{F}_{\mathbf{QCat}^\Delta}(\mathcal{C},\mathcal{E}). \\ = \Delta^1 & & = \mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{D}) \end{array}$$

But again,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1, \mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{D})) &\cong (\mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{D}))_1 \\ &= \{\text{equivalences } \Delta^1 \times \mathcal{C} \to \mathcal{E}\}. \end{aligned}$$

An equivalence $\Delta^1 \times \mathcal{C} \to \mathcal{E}$ witnesses $G \circ F \simeq H$, where $H\colon \mathcal{C} \to \mathcal{E}$ is a functor corresponding to

$$\{0\} \to \Delta^1 \to \mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{E}),$$

and $G \circ F\colon \mathcal{C} \to \mathcal{E}$ the functor corresponding to

$$\{1\} \to \Delta^1 \to \mathrm{Core}\ \mathrm{F}(\mathcal{C},\mathcal{E}).$$

We have the slogan:

Compositions in $\mathbf{Cat}_\infty$ *are compositions in* **sSet** *up to equivalence of functors.*

Similarly in **An**: 0-simplices are Kan complexes, 1-simplices are functors between them and 2-simplices are homotopies (notice the difference in definitions of $\mathrm{F}_{\mathbf{Kan}^\Delta}$ and $\mathrm{F}_{\mathbf{QCat}^\Delta}$). So we have the slogan:

Compositions in **An** *are compositions in* **sSet** *up to homotopy of functors.*

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